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Unit 5
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Gateways Part I—Jigsaw

For the first week of the Gateways, you will be split from your team into a new Jigsaw Team of 4 to 6 NEWT scholars. This group will be assigned a single Gateway to work on for a week.

My team is______________________

Goal of This Week

Each Jigsaw Team will be assigned one of the six Gateway topics. This week will be used for each Jigsaw Team to become masters of their respective topics. In the coming weeks, we will return to our original table groups to cover each topic. We will not endeavor to be *teach*ers to our table group, though. We will be guides and safety nets and be able to give feedback and a point of reference. As such, we will spend this week getting together what we need to be that person for our table groups.

Assignment

By Friday, March 9th, you must…

- Share a Google Doc with Mr. Madonna and everyone on your Jigsaw Team.
- Include in the Google Doc
  - An answer key to your Jigsaw Team's Practice section.
  - Four to six "Check for Understanding" questions to ensure scholars at your original table comprehended the readings (with ideal answers).
  - Notes for each question of where a scholar can look in their notes for guidance.
  - Guiding questions (but not leading questions) (with ideal answers).
- Get approval for the material you turn in.

Grading

- You will earn up to 60 points this week.
- Attendance is key to the success of this unit and you will earn 4 points for each day in attendance (an absence can be made up in office hours).
- The Google Doc will be a group grade of 30 points.
  - 10 points for sharing the Google Doc before Tuesday, March 6th at 8am.
  - 10 points for the answer key
  - 10 points for the CFUs
  - 10 points for guiding questions.
  - All points are assigned as a binary and require approval before March 9th.
  - There are no makeups for this assignment.

Gateways Part II—Big Rocks

A Big Rock on integrals will drop starting Tuesday, March 6th. This will be worth 50 points. A scholars Big Rock Grade B is a function of the average of the best three quizzes, a. $B(a) = 50 + 10a$.

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Gateways Part III—The Actual Gates

Starting March 12th, 2018, we will begin gateways in full as table groups. Each table group will be responsible to set their own homework, ensure they move along as a steady pace, and reach mastery as a table.

Goal of the Gateways

We will cover a total of six topics. Each topic is a culmination of topics from our class, but this is where we fully appreciate the breadth of NEWT Calculus. Each gateway contains a homework assignment. One person in each group has the answer key and will be the person to decide that the group is ready to take the quiz. In order to pass each gateway, every person in the group must achieve 80% or higher on the Gateway Quiz. No one in a group can take the next gateway until everyone has completed the last one.

Grading

- Each Quiz is worth 30 points and can be completed in ~30 minutes.
- An attendance grade will be worth 5 points per week (absences can be made up in office hours within one week)
- Groups can earn 2 points extra credit through 100% attendance to office hours.

What If My Group Doesn't Have 6 People In It?

Many table groups won't have six people in it. In these cases, they will be equipped with the artifacts from the first week for the subjects where they do not have an expert for.

Due Dates

- Gateway Alfa 3/16
- Gateway Bravo 3/23
- Gateway Charlie and Delta 4/6
- Gateway Echo and Foxtrot 4/12
- Note: Spring Break starts after school 3/23

Table of Contents

What If We Finish Early?

There's a bonus gateway, just like before.

What Comes Next?

We have one last, small unit.

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Gateway Alfa: Turning Points and Points of Inflection

A turning point (or **relative extremum**) occurs where a function changes from increasing to decreasing or vice versa. A point of inflection occurs where the function changes from concave up to concave down.

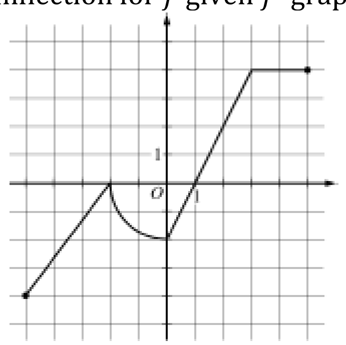
In order for an x-value, let's call it $x = c$, to be a **turning point** of f, it must fulfill two conditions. First, $f'(c)$ must equal 0. From there, we have two options. We can look at $f''(c)$, and as long as this is non-zero, we can classify $x = c$ as a turning point (for a positive second derivative, this will be a minimum, and a negative second derivative, this will be a maximum). The second test is determining if a sign changed occurred in the first derivative at $x = c$.

There is one other option for a turning point… if the $f'(c)$ is **undefined** (but the original function *is* defined), and $x = c$ also satisfies one of the other two conditions.

A **point of inflection** fits the same conditions as a turning point, but for f''. Therefore, $x = c$ is a point of inflection if and only if $f''(c) = 0$ and f'' has a sign change at $x = c$.

Graphical

Find and classify all turning points and points of inflection for f given f' graphed below.



Answer:

f has a turning point at $x = 1$. At $x = 1$, f has a minimum, because f' changes from negative to positive, so f is changing from decreasing to increasing.

f has a point of inflection at $x = -2$ and 0. Because these are turning points of f', this is where its derivative f'' both equals zero and changes sign.

Even though $f'(-2) = 0$, f' does not have a sign change, so not a turning point.

Analytical

Find and classify all turning points and points of inflection for f given

$$f(x) = -3 + 6x^2 - 2x^3$$

Answer:

We first need to find f' and f'', which here only requires use of the power rule

$$f'(x) = 12x - 6x^2$$
$$f''(x) = 12 - 12x$$

We then need to find the zeroes of each.

$$f'(x) = 0 \text{ at } x = 0, 2$$
$$f''(x) = 0 \text{ at } x = 1$$

We can test the zeroes of f' by plugging them into f'' and seeing:

$$f''(0) = 12\ (positive)$$
$$f''(2) = -12\ (negative)$$

We can test for a sign change in f'' by making a number line: (the "positive" and "negative" where determined by using test values less than and greater than 1)

So! f has a minimum at $x = 0$, a maximum at $x = 2$ and a point of inflection at $x = 1$

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Numerical

Find and classify all turning points and points of inflection for f given selected values of f, f' and f'' on the table below.

x	f	f'	f''
-2	4	-2	-4
4	-3	0	-3
5	-1	0	0
6	0	3	1

Answer:

We can classify $x = 4$ as a maximum as $f'(4) = 0$ and $f''(4) < 0$.

There are a few good non-answers here: $x = 5$ is tempting as a turning point because with numerical displays of a function, we just don't know enough about what is going on around the few points we're given to make any kind of conclusion about the sign change.

Verbal

f is twice-differentiable for its entire domain. f is decreasing on the interval $-2 < x < 5$ and $x > 8$ and increasing for $x < -2$ and $5 < x < 8$. Additionally, f is concave down for $x > 4$ and concave up for $x < 4$. Find and classify all turning points and points of inflection for f

Answer

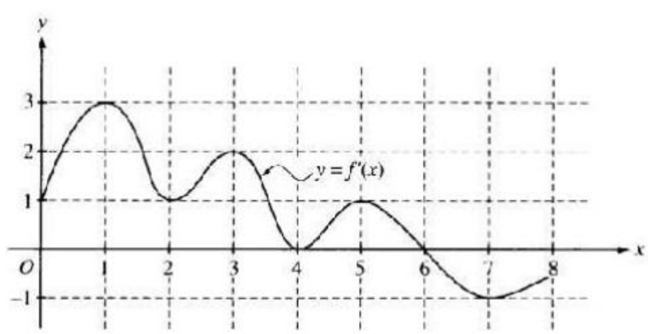
f has a maximum at $x = -2$ (changes from increasing to decreasing), a minimum at $x = 5$ (changes from decreasing to increasing), and a maximum at $x = 8$ (same rationale as $x = -2$).

f has a point of inflection at $x = 4$ due to the concavity change.

1.) Fill out the table…

	Relationship to f	Turning points of f are where…	Points of inflection of f are where…	When positive, f is…
$\int_0^x f(t)dt + f(0)$				
$f(x)$				
$f'(x)$				
$f''(x)$				

Graphical



2.) The graph above shows $f'(x)$ on the closed interval [0,8]. How many points of inflection does f have? Locate and classify all turning points of f.

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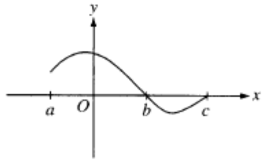
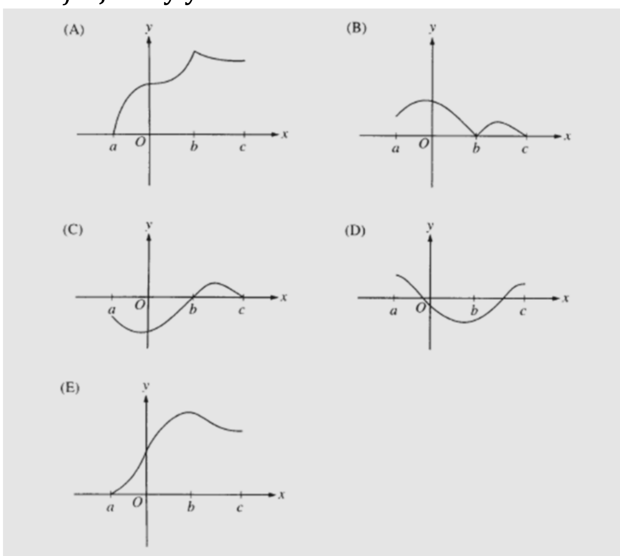
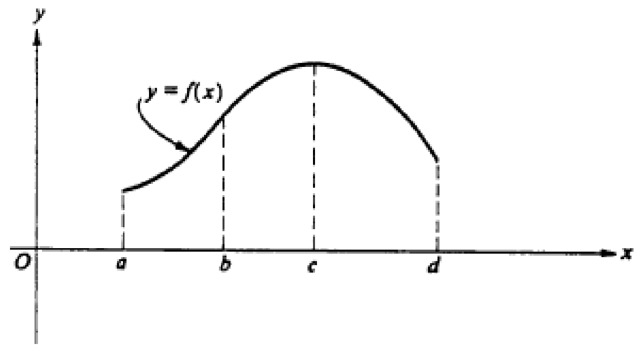


3.) Let $f(x) = \int_a^x h(t)dt$ where h has the graph shown above. Which of the following could be the graph of f? Justify your answer.





4.) The graph of $y = f(x)$ is shown in the figure above. On which of the following intervals is $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} < 0$? Circle all that apply.

I. $a < x < b$
II. $b < x < c$
III. $c < x < d$

Numerical

Use the twice-differentiable functions f and g with selected values are given below for questions #5 and 6.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$	$f''(x)$	$g''(x)$
-2	-5	1	-6	5	4	0
0	1	-7	-1	0	5	-7
1	-1	-3	4	6	6	7
4	2	2	-1	0	-3	3

5.) Locate and classify all turning points of $g(x)$

6.) Located and classify all turning points of $h(x)$ for $h(x) = \frac{1}{2}f(x^2) - \frac{1}{2}x^2$

7.) Let F be a function defined for all real numbers x such that $F'(x) > 0$ and $F''(x) > 0$. Which of the following could be a table of values for F?

a.)

x	$F(x)$
1	-3
2	-4
3	-6
4	-9

b.)

x	$F(x)$
1	-3
2	-1
3	3
4	19

c.)

x	$F(x)$
1	-3
2	0
3	3
4	6

d.)

x	$F(x)$
1	-3
2	5
3	11
4	13

e.)

x	$F(x)$
1	-3
2	-4
3	-3
4	0

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Analytical

8.) Let f be the function defined by $f(x) = 12x^{\frac{2}{3}} - 4x$. Find the intervals on which f is increasing.

9.) A particle moves along the x-axis so that at time $t > 0$ its position is given by $x(t) = 12e^{-t}\sin t$. What is the first time t at which the velocity of the particle is zero? Is this a maximum or a minimum?

10.) A particle moves along the x-axis. For $0 \leq t \leq 8$, the velocity of a particle P at time t is given by $v_p = t^2 - 8t + 15$. Find all times t when the particle changes direction.

Verbal

Let f be a function that is continuous on the interval $[0,4)$. The function f is twice differentiable except at $x = 2$. The function f and its derivatives have the properties indicated in the table below.

x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$	3	$3 < x < 4$
$f(x)$	-1	Negative	0	Positive	2	Positive	0	Negative
$f'(x)$	4	Positive	0	Positive	DNE	Negative	-3	Negative
$f''(x)$	-2	Negative	0	Positive	DNE	Negative	0	Positive

11.) For $0 < x < 4$, find all values of x at which f has a relative extremum. Determine whether f has a relative maximum or a relative minimum at each of these values. Justify your answer. Find any coordinates of any inflection points on the graph of f. Justify your answer.

12.) For $x > 0$, f is a function such that $f'(x) = \frac{\ln x}{x}$ and $f''(x) = \frac{1-\ln x}{x^2}$. Which of the following is true?

a. f is decreasing for $x > 1$, and the graph of f is concave down for $x > e$
b. f is decreasing for $x > 1$, and the graph of f is concave up for $x > e$
c. f is increasing for $x > 1$, and the graph of f is concave down for $x > e$
d. f is increasing for $x > 1$, and the graph of f is concave up for $x > e$
e. f is increasing for $0 < x < e$, and the graph of f is concave down for $0 < x < e^{3/2}$

13.) A differentiable function f has the property that $f'(x) \leq 3$ for $1 \leq x \leq 8$ and $f(5) = 6$. Which of the following could be true (circle all that apply)?

I. $f(2) = 0$
II. $f(6) = -2$
III. $f(7) = 13$

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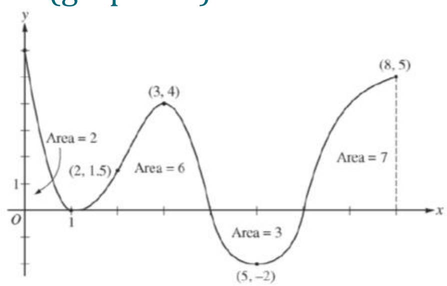
Gateway Bravo: Extreme Values

When differentiating between **relative** and **absolute** extrema, the difference only really matters when dealing with functions with **constraints** or **bounds**. A **relative extremum**, also called a **local extremum**, is a turning point that may be higher or lower that everything surrounding it, but not necessarily the highest and lowest point period. An **absolute maximum** is, as it sounds, the highest point on that graph. Likewise, an **absolute minimum** is the lowest point on the graph. Discussing these will entail a look at mostly real-life contexts, as those are the functions that will have constraints.

To find extreme values of a function f, we must first locate all x-values of all turning points of f. We then locate all endpoints of f. We then compare the y-values at the turning points and endpoints. The biggest one will be the absolute maximum. The smallest one will be the absolute minimum.

Example 1 (graphical)



Graph of f'

The figure to the left shows the graph of f', the derivative of a twice-differentiable function f, on the closed interval $0 \leq x \leq 8$. The graph of f' has horizontal tangent lines at $x = 1, x = 3$ and $x = 5$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(8) = 4$.

Answer

The turning points of f are located at $= 4$ and $x = 6$ (this is where f' changes sign). The endpoint of f are given in the question as $x = 0$ and $x = 8$. So we make a table:

x	$f(x)$
0	
4	
6	
8	4

We know $f(8)$ based on the question, but in order to fill out the rest of the $f(x)$ column, we must use the definition of integral as <u>net change</u> and as <u>area</u>. For example, $\int_6^8 f'(x)dx$ both equals the area between 6 and 8 (i.e., -3) and $f(8) - f(6)$. We can set up and solve an expression:

$$-3 = f(8) - f(6)$$
$$-3 = 4 - f(6)$$
$$f(6) = 7$$

We do this for the rest of $f(x)$ and get the table:

x	$f(x)$
0	-2
4	1
6	7
8	4

Therefore, at $x = 0$, f has an absolute minimum with the value -2 and at $x = 6$, f has an absolute maximum with a value of 7.

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Example 2

The rate at which people enter an auditorium for a wrock concert is modeled by the function R given by $R(t) = 1380t^2 - 675t^3$ for $0 \le t \le 2$ hours; $R(t)$ is measured in people per hour. What is the maximum rate at which people are entering the auditorium?

Answer

Though it doesn't say, we can interpret "maximum" here to mean absolute maximum because of the closed interval. So we need to find all turning points of R and compare the values at these points with the values at the endpoints.

$$\boldsymbol{R'(t) = 2760t - 2025t^2}$$

$$\boldsymbol{R'(t) = 0 \text{ at } t = 1.363}$$

x	$f(x)$
0	0
1.363	854.527
2	2

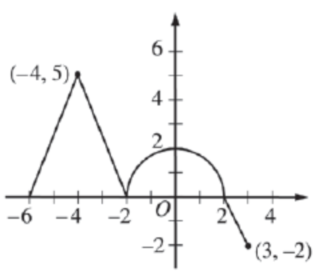
(All table values were found by direct substitution into $R(t)$.) R it at its maximum at $t = 1.363$.

Practice

1.) The function g is given by $g(x) = 4x^3 + 3x^2 - 6x + 1$. What is the absolute minimum of g on the interval $[-2,1]$?

2.) The function f is defined for all x in the closed interval $[a, b]$. If f does not attain a maximum value on $[a, b]$, which of the following must be true?

a. f is not continuous on $[a, b]$
b. f is not bounded on $[a, b]$
c. f does not attain a minimum value on $[a, b]$
d. The graph of f has a vertical asymptote in the interval $[a, b]$
e. The equation $f'(x) = 0$ does not have a solution in the interval $[a, b]$



Graph of f

3.) The graph of the continuous function f, consisting of three line seqments and a semicircle, is shown above. Let g be the function given by $g(x) = \int_{-2}^{x} f(t)dt$. Determine the absolute maximum and minimun of g on the interval $[-6,3]$.

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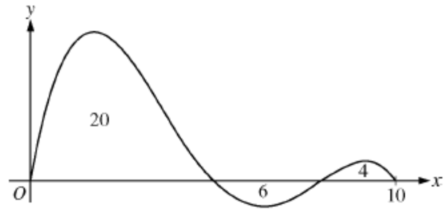
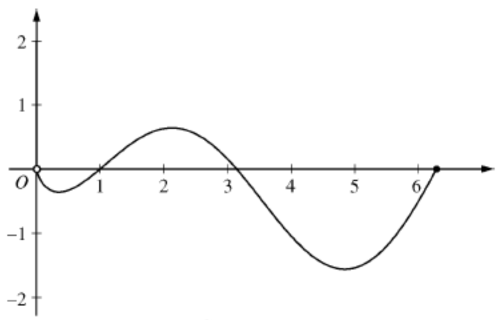


Graph of f'

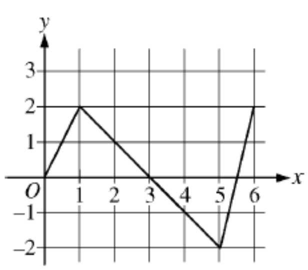
4.) The graph of f', the derivative of the function f, is shown above for $0 \leq x \leq 10$. The areas of the regions between the graph of f' and the x-axis are 20, 6 and 4 respectively. If $f(0) = 2)$, what is the maximum value of f on the closed interval $0 \leq x \leq 10$.



Graph of f

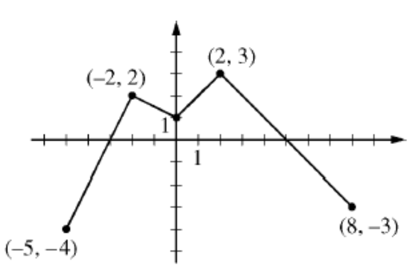
5.) Let f be the function given by $f(x) = (\ln x)(\sin x)$. The figure above shows the graph of f for $0 < x \leq 2\pi$. The function g is defined by $g(x) = \int_1^x f(t)dt$ for $0 < x \leq 2\pi$. Find the value of x at which g has an absolute minimum. Justify your answer.

6.) Let $g(x) = xe^{-x} + be^{-x}$, where b is a positive constant. For what positive value of b does g have an absolute maximum at $x = \frac{2}{3}$? Justify your answer.



Graph of f'

7.) For $0 \leq x \leq 6$, the graph of f', the derivative of f, is piecewise linear as shown above. If $f(0) = 1$, what is the maximum value of f on the interval?



Graph of f

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8.) The continuous function f is defined on the interval $-5 \leq x \leq 8$. The graph of f, which consists of four line segments, is shown in the figure above. Let g be the function given by $g(x) = 2x + \int_{-2}^{x} f(t)dt$. Locate the absolute maximum and minimum.

9.) Particle X moves along the positive x-axis so that its position at time $t \geq 0$ is given by $x(t) = 5t^3 - 9t^2 + 7$. At what time $t \geq 0$ is the particle farthest to the left? Justify your answer.

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Gateway Charlie: Existence Theorems (IVT and MVT)

Definitions

An **open interval** excludes the endpoints of an interval. A **closed interval** includes the endpoints. For example, $2 < x < 5$ is an open interval because it doesn't include 2 or 5. $2 \le x \le 5$ is a closed interval.

A function is **continuous** on an open interval $a < x < b$ if for every x-value in the interval, called c, $f(c)$ is defined and equal to $\lim_{x \to c} f(x)$. Practically, this is a function with no gaps or breaks on the interval. A function is continuous on a closed interval $a \le x \le b$ if f is continuous at $x = a$, $x = b$ and the open interval $a < x < b$.

A function is **differentiable** on an open interval $a < x < b$ if for every x-value in the interval, called c, $f'(c)$ is defined. Practically, this is a function with no sharp turns or vertical tangent lines. Any function that is **differentiable** is also continuous. If a function is not continuous at a point, it is also not differentiable.

A function is **twice-differentiable** on an open interval $a < x < b$ if f is differentiable on this interval *and* f' is differentiable on the same interval. Basically, at any point, the derivative and second derivative exists.

Intermediate Value Theorem

The **intermediate value theorem (IVT)** states the obvious: for a continuous function, if the graph travels from some y-coordinate c to another y-coordinate d, it must pass every point in between. There was a word in the explanation that we might have glossed over, but let's stress this word: **continuous.** We can imagine a whole host of discontinuous functions for which this isn't true.

Mean Value Theorem

The **mean value theorem (MVT)** says that for a differentiable function f on the interval $a \le x \le b$, there must exist some value c inside the interval such that $f'(c)$ is equal to the average rate of change over the interval.

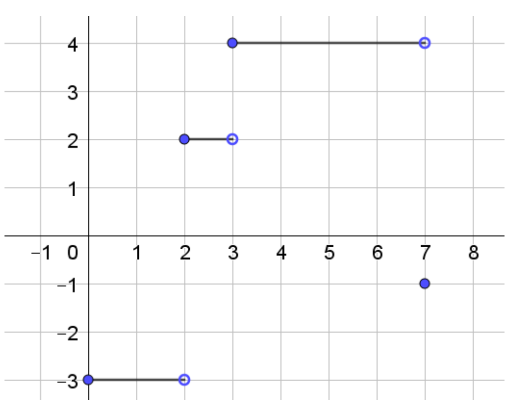
IVT Ex. 1

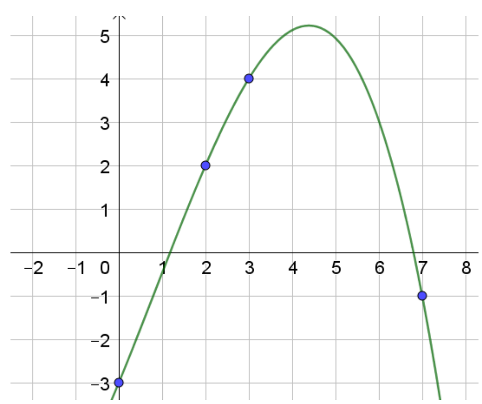
x	0	2	3	7
$f(x)$	-3	2	4	-1

The table above gives selected values for the function f on the interval $0 \le x \le 7$. At least how many x-intercepts must f have?

Answer:

Short answer… there are no guaranteed x-intercepts because we don't know if this function is continuous. Because of the limited information, either of the two graphs could be a graph of f. For the graph on the right, it is clear why continuity is needed in order to draw this conclusion.





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IVT Ex. 2

x	0	2	3	7
$f(x)$	-3	2	4	-1

The table above gives selected values for the twice-differentiable function f on the interval $0 \leq x \leq 7$. At least how many x-intercepts must f have?

Answer

This is one that matters, because f is continuous. We know it's continuous because f is differentiable, which means it also must be continuous.

f is differentiable and therefore continuous therefore we can use IVT. At an x-intercept, $f(c) = 0$. Because $f(0) < 0 < f(2)$, f must have an intercept between 0 and 2. Because $f(3) > 0 > f(7)$, f must have an intercept between 3 and 7. Therefore f must have <u>at least two</u> intercepts on this interval.

MVT Verbal Example

Engineers at a testing a classic Chevelle at a 1 mile proving ground—a straight, flat track where you can check for the top speed of the car. Two sensors are set up at the quarter mile and at 1 mile. The car reaches the first sensor after 13.7 seconds and the second at 30 seconds. The velocity readout at each sensor is 103 miles per hour and 140 miles per hour. The goal for the Chevelle was to reach 150 miles per hour. Did the engineers reach this goal?

Answer

It's important to note that position, $x(t)$, of an object in nature are both going to be differentiable functions, therefore we can use the MVT. We know two points on the position function of the Chevelle, $x(13.7) = 0.25$ and $x(30) = 1$. Therefore the average rate of change between these two points is:

$$\frac{1 - 0.25}{30 - 13.7} = 0.0433 \text{ mi/s}$$

Because the units of speed we want are miles per hour, quick conversion will result in:

$$\frac{0.0433 \text{ mi}}{1 \text{ s}} \times \frac{3600 \text{ s}}{1 \text{ hr}} = 156.069 \text{ mi/hr}$$

Yes! By the mean value theorem, for the differentiable function $x(t)$, at some point between 13.7 s and 30 s, the car's velocity, or $x'(t)$ must equal 156.069 mi/hr, its average rate of change. Therefore it exceeded the goal.

MVT Ex. 2

The function g is continuous on the closed interval $[1,4]$ with $g(1) = 5$ and $g(4) = 8$. Of the following conditions, which would guarantee that there is a number c in the open interval $(1,4)$ where $g'(c) = 1$?

a. g is increasing on the closed interval $[1,4]$
b. g is differentiable on the open interval $(1,4)$
c. g has a maximum value on the closed interval $[1,4]$
d. The graph of g has at least one horizontal tangent in the open interval $(1,4)$

Answer

B. This is a straight forward application of the MVT. The average rate of change on the interval $[1, 4]$ is given by:

$$\frac{g(4) - g(1)}{4 - 1} = \frac{8 - 5}{4 - 1} = \frac{3}{3} = 1$$

But in order to conclude that there some number c such that $g'(c) = 1$, we need to know that g is <u>differentiable</u>. *Note, the problem established g was continuous, but that is not enough to conclude continuous. Differentiable implies continuous, but not vice versa.*

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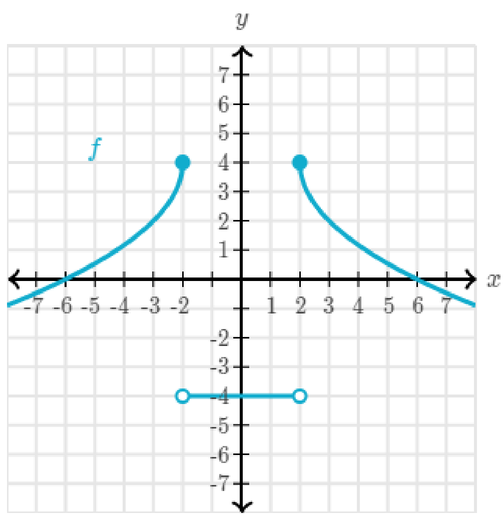
t	0	10	40	60
$B(t)$ (meters)	100	136	9	49
$V(t)$ (meters per second)	2.0	2.3	2.5	4.6

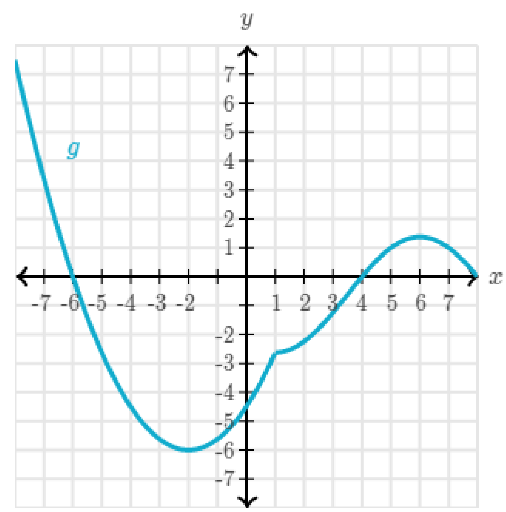
1.) Ben rides a unicycle back and forth along a straight east-west track. The twice-differentiable function B models Bens' position on the track, measured in meters from the western end of the track, at time t, measured in seconds from the start of the ride. The table above gives values for $B(t)$ and Ben's velocity, $v(t)$, measured in meters per second, at selected times t. For $40 \leq t \leq 60$, must there be a time where Ben's velocity is 2 meters per second? Justify your answer.

x	3	4	5	6
$g(x)$	-17	-13	-13	-15

2.) The table above gives selected values of $g(x)$. Which condition would allow you to conclude that there exists a solution to $g(x) = -15$ in the interval $[3,5]$?

a. g is differentiable over the closed interval $[4,5]$
b. g is continuous over the open interval $(3,6)$ and at $x = 3$
c. g is continuous over the open interval $(3,5)$ and at $x = 5$





3.) For f graphed above, does the Mean Value Theorem apply to f over the interval $[2,5]$?

4.) For g graphed above, does the Mean Value Theorem apply to g over the interval $[-2,6]$?

x	9	10	11	12
$g(x)$	18	17	14	17

5.) The table above gives selected values for $h(x)$. Mitya said that since $\frac{h(12)-h(11)}{12-11} = 3$, there must be a number c in the interval $[11,12]$ for which $h'(c) = 3$. Which condition makes Mitya's claim true?

a. $\lim_{x \to 11.5} h'(x) = 3$
b. h is differentiable over the open interval $(9,12)$ and continuous over the closed interval $[9,12]$.
c. h is continuous over the closed interval $[11,12]$ and differentiable at $x = 11$ and $x = 12$
d. h is differentiable over the open interval $(11,12)$ and continuous at $x = 12$.

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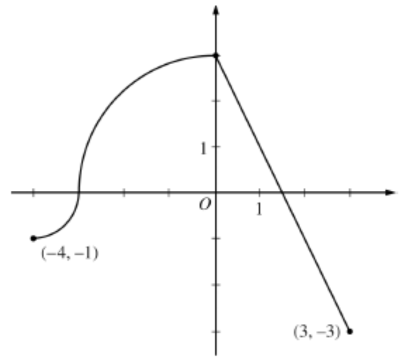

Graph of f

6.) The continuous function f is defined on the interval $-4 \leq x \leq 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above. Let $g(x) = 2x + \int_0^x f(t)dt$. Find the average rate of change on the interval $-4 \leq x \leq 3$. There is no point c, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$

7.) The twice-differentiable function f is defined for all real numbers x. Values of f and f' for various values of x are given in the table above. Explain where there must be a value c, for $-1 < c < 1$, such that $f''(c) = 0$.

x	$g(x)$	$g'(x)$
-5	10	-3
-4	5	-1
-3	2	4
-2	3	1
-1	1	-2
0	0	-3

8.) Let g be a differentiable function. The table above gives values of g and its derivative g' at selected values of x. Is there a number c in the closed interval $[-5, -3]$ such that $g'(c) = -2$? Justify your answer. Is there a number k in the closed interval $[-5, -3]$ such that $g'(k) = -4$? Justify your answer.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

9.) The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x) - 6$. Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$. Explain where there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

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Gateway Delta: Definition of Derivative and L'Hôspital's Rule

Limit Definitions of Derivative

The derivative can be found by taking the slope of a secant line between two points and finding the limit as those two points converge. We therefore get two limit definitions of derivative:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

These are where our derivative rules (power rule, chain rule, product rule, at al) are derived from. But because we have determine our rules, we no longer need to use limits to find derivatives. That said, a common question on the AP test might be:

Example

Determine the limit:

$$\lim_{x \to \pi} \frac{\cos x - 1}{x - \pi}$$

Answer

We needn't evaluate the limit here! We instead need to <u>recognize</u> that this in fact the limit definition of the derivative. We then realize this question is <u>really</u> asking us to determine $f'(c)$. In this case, $f(x) = \cos x$ and $c = \pi$. Therefore $f'(x) = -\sin x$ and $f'(c) = -\sin \pi = -0 = 0$.

Horizontal Asymptotes and Limits at Infinity

Consider the limit:

$$\lim_{x \to -\infty} \left(\frac{3x^2 - 2x + 4}{x^3 + 5x^2 - 4x + 3} \right)$$

This limit is effectively asking for the **end behavior** of the function inside the limit. Remember our precalculus, we know we have some options for end behavior—the function diverges to positive or negative infinity or has a horizontal asymptote. Therefore rules for determining asymptotes still exist. Note, though that function can have two asymptotes, one on the left and one on the right. In the example above, because the degree of the denominator is greater than the degree of the numerator, the function inside the limit has a horizontal asymptote at $y = 0$.

Example

$$\lim_{x \to -\infty} \left(\frac{2 + e^x}{1 + e^x} \right)$$

Answer

Our asymptote rules don't apply here as the numerator and denominator aren't polynomials. But realizing that the numerator approaches 2 as $x \to -\infty$ and the denominator approaches 1 as $x \to -\infty$, therefore we can conclude the limit approaches 2.

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L'Hôspital's Rule

Limits are typically kind of easy. By using direct substitution, we have one of three cases—a result in a number value, a result where the numerator of the function is nonzero and the denominator is nonzero, or a result where both the numerator and denominator result in zero. With the first result, that number result is the limit. The second result indicates a vertical asymptote and an infinite discontinuity. The third result is the one we care about and is called **indeterminate form**. In the past, we've used algebra to manipulate expressions to evaluate limits. We're going to use something way more awesome here, called **L'Hôspital's Rule** (also sometimes spelled L'Hôpital's rule). It says this:

$$\text{If } \lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0, \text{then } \lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

Additionally:

$$\text{If } \lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \pm\infty, \text{then } \lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

In summary, this says that if a limit evaluates to indeterminate form (either $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$), you can take the derivative of the numerator and denominator individually and then evaluate the derivative again through direct substitution with the new function.

Example

Looking at our example from earlier that used the definition of derivative

$$\lim_{x\to\pi}\frac{\cos x - 1}{x - \pi}$$

Answer

Because $\frac{\cos\pi - 1}{\pi - \pi}$ we can use L'Hôspital's rule here too!

$$\lim_{x\to\pi}\frac{\cos x - 1}{x - \pi} = \lim_{x\to\pi}\frac{\frac{d}{dx}[\cos x - 1]}{\frac{d}{dx}[x - \pi]} = \lim_{x\to\pi}\frac{-\sin x}{1} = -\sin\pi = 0$$

This is the same result as above. Though we can use the earlier skill of recognizing that this is the definition of derivative, both ways work.

Note: This can be confusing because the process of finding the derivative of the numerator and denominator looks like a common error that is easy to make when learning the quotient rule! But this is something very different. We are not trying to find the derivative of a function, but rather evaluating a limit.

Example

This is the same function as the second example above, but not the same problem as the limit changed from $-\infty$ to ∞

$$\lim_{x\to\infty}\left(\frac{2 + e^x}{1 + e^x}\right)$$

Answer

Through direct substitution, we get $\frac{2+e^\infty}{1+e^\infty} = \frac{2+\infty}{1+\infty} = \frac{\infty}{\infty}$ (if you'll pardon the sloppy use of infinity here). This makes this a candidate for L'Hôspital's Rule.

$$\lim_{x\to\infty}\left(\frac{2 + e^x}{1 + e^x}\right) = \lim_{x\to\infty}\left(\frac{\frac{d}{dx}[2 + e^x]}{\frac{d}{dx}[1 + e^x]}\right) = \lim_{x\to\infty}\left(\frac{e^x}{e^x}\right) = \lim_{x\to\infty} 1 = 1$$

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Practice

1.) Evaluate

$$\lim_{x\to 2}\frac{x^2+x-6}{x^2-4}$$

2.) For which of the following does $\lim_{x\to\infty} f(x) = 0$? (Circle all that apply)

I. $f(x) = \frac{x^2}{e^x}$

II. $f(x) = \frac{e^x}{e^{2x}}$

III. $f(x) = \frac{x^9}{e^x}$

IV. $f(x) = \frac{x^{99}}{e^x}$

3.) Let f be the function given by $(x^2-2x-1)e^x$. Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

4.) Evaluate

$$\lim_{h\to 0}\frac{e^{2+h}-e^2}{h}$$

5.) Evaluate

$$\lim_{x\to 3}\frac{\int_3^x e^{t^2}\,dt}{x-3}$$

6.) Evaluate

$$\lim_{x\to\infty}\left(\frac{x^3}{e^{3x}}\right)$$

7.) Let $g(x) = xe^{-x} + be^{-x}$. Find $\lim_{x\to\infty} g(x)$

8.) Evaluate

$$\lim_{x\to 2}\frac{\ln(x+3)-\ln(5)}{x-2}$$

9.) The graph of the even function $y = f(x)$ contains 4 line segments as shown above. Which of the following statements about f is false?

a. $\lim_{x\to 0}\left(f(x)-f(0)\right) = 0$

b. $\lim_{x\to 0}\frac{f(x)-f(0)}{x} = 0$

c. $\lim_{x\to 0}\frac{f(x)-f(-x)}{2x} = 0$

d. $\lim_{x\to 2}\frac{f(x)-f(2)}{x-2} = 1$

e. $\lim_{x\to 2}\frac{f(x)-f(3)}{x-3}$ does not exist.

10.) Let f be the function defined by $f(x) = \frac{(3x+8)(5-4x)}{(2x+1)^2}$. Determine the horizontal asymptotes to the graph of f.

11.) Let f be the function defined by $f(x) = \frac{\sqrt{x^4+2x^2+8}}{x^2+9}$. Determine the horizontal asymptotes to the graph of f.

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Gateway Echo: Calculus with Logs and Stealth Review of Tangent Lines

You, Logs, and You

Let's recall what a logarithm is from our Algebra 2 experience (eep!) before we go too far. First of all, logarithms (sometimes called "logs" if you're cool) are the inverse operation of exponentiation. Namely:

$$b^x = y \Leftrightarrow \log_b y = x$$

We've seen that "log base 10 of x" is written as $\log x$ and "log base e of x" is written $\ln x$. For reasons that we have already sort of observed, e is a very important number in Calculus, and the only log we will deal with is the natural log (to wit, we will also deal with e as the only exponential base). In fact it is so common, that we will likely just read $\ln x$ as "log of x" instead of "natural log of x."

Algebra Review

Just like exponents have special rules (e.g. $b^n \cdot b^m = b^{n+m}$ among others), logs have rules!

$\ln e^m = m$ $\qquad$ $\ln 1 = 0$ $\qquad$ $\ln x$ is undefined for $x \leq 0$

$\ln nm = \ln n + \ln m$ $\qquad$ $\ln\left(\frac{n}{m}\right) = \ln n - \ln m$ $\qquad$ $\ln n^m = m \ln n$

Calculus with Logs

As mentioned, e is of huge importance in calculus because of this property:

$$\text{If } y = e^x, \text{then } y' = y = e^x$$

A function whose derivative is itself is a big deal. So what's the derivative of $\ln x$? Well…

Proof $\frac{d}{dx}[\ln x] = \frac{1}{x}$

$y = \ln x$	Let's start with the function we're trying to differentiate
$e^y = x$	We're rewriting the log as an exponential
$\frac{d}{dx}[e^y = x]$	We'll differentiate this because we know how to
$e^y \frac{dy}{dx} = 1$	The $\frac{dy}{dx}$ comes from implicit differentiation!
$\frac{dy}{dx} = \frac{1}{e^y}$	Getting $\frac{dy}{dx}$ by itself… wait, doesn't $e^y = x$?
$\frac{dy}{dx} = \frac{1}{x}$	Quick substitution!

With this result, we can also state:

$$\int \frac{1}{x} dx = \ln|x| + C$$

(Yes there are absolute value bars around the inside of the logarithm. This is due to the fact that $\frac{1}{x}$ has a domain of all real numbers such that $x \neq 0$ but $\ln x$ is only defined for positive x values).

It should be noted that $\int \ln x \, dx$ is not an easy integral to do and not something we should memorize, but the antiderivative of $\frac{1}{x}$ and the derivative of $\ln x$ absolutely are.

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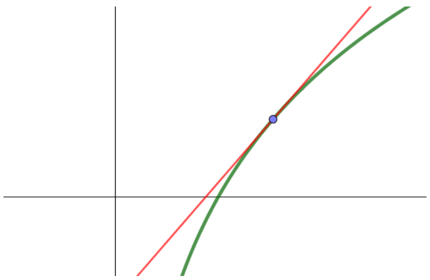
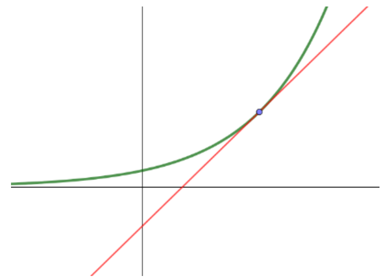
Tangent Lines in 200 Words or Less

Internalize this… a line tangent to a function shares two attributes with the function: the y-value and the slope of the function at the point of tangency. We can write the general form of a line tangent to a function f at the point $x = c$ to be:

$$y = f'(c)(x - c) + f(c)$$

We can use the tangent line to approximate values of the function close to the point of tangency, especially useful in times when working sans calculator. As the approximation will never be 100% right, we can also determine if this represents an **overestimate** or an **underestimate.** Consider the two graphs below:





In the picture on the left, the tangent line would overestimate the value of the function. On the right, underestimate. **The concavity determines this**. For a graph that is concave down at the point of tangency (that is, the second derivative is negative as in the graph on the right), the tangent line is above the graph and would serve as an overestimate. And vice versa for concave up graphs, as in the graph on the left.

Example

$$\frac{d}{dx}[\ln(x^2 + 5)]$$

Answer

Using chain rule, $u = x^2 + 5, f(u) = \ln x, f'(u) = \frac{1}{u}, u' = 2x$ Thefore:

$$\frac{d}{dx}[\ln(x^2 + 5)] = f'(u)u' = \frac{2x}{x^2 + 5}$$

Example

$$\int \frac{4}{x} dx$$

Answer

This is a straight application of the antiderivative of $\frac{1}{x}$ once your recognize that

$$\int \frac{4}{x} dx = 4 \int \frac{1}{x} dx = 4 \ln |x| + C$$

Example

$$\int \frac{x}{x^2 + 1} dx$$

Answer

Let $u = x^2 + 1$. This means $du = 2x dx$ and therefore $\frac{1}{2} du = x dx$

$$\int \frac{x}{x^2 + 1} dx = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + C$$

After undoing the substitution, we get a final answer of $\frac{1}{2}\ln|x^2+1|+C$

Example

$$\frac{d}{dx}\left[\ln\left(e^3x^4\sqrt{x^2-1}\right)\right]$$

Answer

Though this examples reads as incredibly intimidating at first, fear not! We can use log rules to greatly simplify the expression first.

$\ln\left(e^3x^4\sqrt{x^2-1}\right)$	**Original expression**
$\ln e^3+\ln x^4+\ln\left(\sqrt{x^2-1}\right)$	**Expanding with addition rule**
$3+4\ln x+\frac{1}{2}\ln(x^2-1)$	**Remember $\ln e^3=3$ because logs are the inverse operations of exponentiation.**

We now have a much simply derivative (though the last term still requires chain rule

$$\frac{d}{dx}\left[3+4\ln x+\frac{1}{2}\ln(x^2-1)\right]$$

$$\frac{d}{dx}[3]+\frac{d}{dx}[4\ln x]+\frac{d}{dx}\left[\frac{1}{2}\ln(x^2-1)\right]$$

$$0+\frac{4}{x}+\frac{1}{2}\cdot\frac{1}{x^2-1}\cdot 2x$$

$$\frac{4}{x}+\frac{x}{x^2-1}$$

Example

$$\int\frac{dx}{x\ln x}$$

Answer

Let $u=\ln x$. Therefore $du=\frac{1}{x}dx$. We can then substitute:

$$\int\frac{du}{u}=\ln|u|+C=\ln|\ln x|+C$$

Yes, that is a confusing result… a log of a log, but that's the answer.

Example

Use the tangent line at $x=-1$ to approximate $f(-1.1)$ for $f(x)=x^2-\ln\left(\frac{1}{x+2}\right)$. Is this estimate an over- or underestimate?

Answer

We have to do a few things to answer this question.

1.) **Find the y-value at $x=-1$**
2.) **Find the first derivative, substitute $x=-1$**
3.) **Find the second derivative, substitute $x=-1$**

$f(-1)$ is easy enough

$$f(-1)=(-1)^2-\ln\left(\frac{1}{(-1)+2}\right)=1+\ln 1=1$$

f' will require us to rewrite…

$$\boldsymbol{x^2 - \ln\left(\frac{1}{x+2}\right) = x^2 - (\ln 1 - \ln(x+2)) = x^2 + \ln(x+2)}$$

$$\boldsymbol{f'(x) = 2x + \frac{1}{x+2}}$$

$$\boldsymbol{f'(-1) = -2 + \frac{1}{(-1)+2} = -2 + 1 = -1}$$

$$\boldsymbol{f''(x) = 2 - \frac{1}{(x+2)^2}}$$

$$\boldsymbol{f''(-1) = 2 - \frac{1}{((-1)^2+2)^2} = 2 - 1 = 1}$$

Therefore the equation of the line tangent at $x = -1$ is:

$$\boldsymbol{y = f'(-1)(x+1) - f(-1)}$$

$$\boldsymbol{y = -(x+1) - 1}$$

Substituting $x = -1.1$

$$\boldsymbol{y \approx -(-1.1+1) - 1 = 0.1 - 1 = -0.9}$$

So $f'(-1.1) \approx -0.9$. Because $f''(-1) > 0$, f is concave up at $x = -1$ so this approximation is an underestimate.

Practice

1.) Differentiate

$$\frac{d}{dx}\left[\ln\left(x^3\sqrt{x+1}\right)\right]$$

2.) Differentiate.

$$\frac{d}{dx}[\cos(\ln x)]$$

3.) Evaluate

$$\frac{d}{dx}\left[\ln\left(\frac{x}{(x+3)^2}\right)\right]$$

4.) Differentiate:

$$\frac{d}{dx}\left[\frac{\sqrt{4+x^2}}{x}\right]$$

5.) Differentiate

$$\frac{d}{dx}\left[\ln\left(x + \sqrt{4+x^2}\right)\right]$$

6.) Integrate

$$\int \frac{5}{x}\,dx$$

7.) Evaluate

$$\int_2^4 \frac{dx}{5-3x}$$

8.) If $\ln(2x+y) = x+1$ then $\frac{dy}{dx} =?$

9.) Integrate:

$$\int \frac{e^x}{1-e^x}\,dx$$

10.) Integrate:

$$\int \frac{3}{2x}\,dx$$

11.) For $x > 0$,

$$\frac{d}{dx}\left(\int_0^{2x} \ln(t^3+1)\,dt\right) =$$

a. $\ln(x^3+1)$
b. $\ln(8x^3+1)$
c. $2\ln(x^3+1)$
d. $2\ln(8x^3+1)$
e. $24x^2\ln(8x^3+1)$

12.) What is the slope of the line tangent to the graph of $y = \ln(2x)$ at $x = 4$?

13.) Write the equation of the line tangent to the function f below at $x = -1$. Is this an overestimate or an underestimate?

$$f(x) = \frac{1}{x} - \ln(x+2)$$

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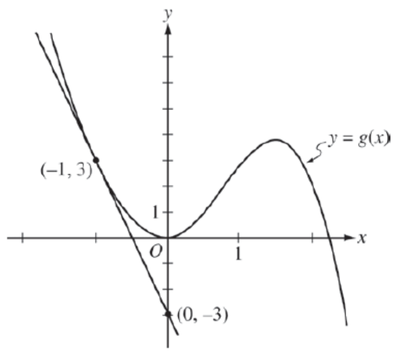
14.) A curve is defined by the equation $x^2 + y^2 = 25$. Show that $\frac{dy}{dx} = -\frac{x}{y}$ and that $\frac{d^2y}{dx^2} = \frac{-x^2-y^2}{y^3}$. Write the equation of the line tangent to the curve in the fourth quadrant at $x = 2$. Use this tangent line to estimate the y-coordinate of the curve in the fourth quadrant at $x = 2.1$. Is an overestimate or an underestimate?

15.) For $f(4) = 3, f'(4) = -1$ and $f''(4) = 5$, use the line tangent to the function $g(x) = f(x^2)$ at $x = -2$ to approximate $g(-1.99)$. Is this an overestimate or an underestimate?



16.) The figure above shows the graph of the function g and the line tangent to the graph of g at $x = -1$. Let h be the function given by $h(x) = e^x \cdot g(x)$. What is the value of $h'(-1)$?

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Accumulation Functions

> On a certain workday, the rate, in tons per hour, at which unprocessed gravel arrives at a gravel processing plant is modeled by $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$, where t is measured in hours and $0 \le t \le 8$. At the beginning of the workday ($t = 0$), the plan has 500 tons of unprocessed gravel. During the hours of operation, $0 \le t \le 8$, the plant processes gravel at a constant rate of 100 tons per hour.

This question stem sets up two functions, one that is **giving** something to an initial amount, the other is **removing** that same thing. This is what we call an accumulation problem. In general, if the "giving" function has a rate of $G(t)$ and the "removing" function has a rate of $R(t)$, then the amount is accumulating with a rate of $A(t) = G(t) - R(t)$. There are a few characteristics of these problems we note are true in the above example:

- Each function is given to us with different representations. Here, a verbal representation and an analytical representation (equation). Numerical (table) and graphical representations are also possible.
- Each function is given as a rate, meaning if we want the amount given, removed or accumulated, we need an integral.
- Each function is given as a rate, meaning if both functions equal to each other, the amount is not changing in size at that moment, also, the rate of accumulation is 0, meaning a turning point.
- Capital letters are used despite these being rates. In the past, capital letters are often used for antiderivatives, which these are not.
- As we'll see, these problems tend to incorporate every topic ever covered, including those in these gateways.

Let's try an example. With the stem above, we'll answer the questions:

- Find $G'(5)$. Using correct units, interpret your answer in the context of the problem.
- Find the total amount of unprocessed gravel that arrives at the plant during the hours of operation on this workday.
- Is the amount of unprocessed gravel at the plant increasing or decreasing at time $t = 5$ hours? Show the work that leads to your answer.
- What is the maximum amount of unprocessed gravel at the plant during the hours of operation on this workday? Justify your answer?

Answer

Before we jump in, we're not given a name for the removing function, so let's call it $R(t)$, in tons per hour, and based off the verbal description $R(t) = 100$, that is, gravel is removed at a constant rate of 100. Note this a positive value. Our accumulation function $A(t) = G(t) - R(t)$ measures the rate of accumulation in tons of gravel per hour.

- $G'(5) = -24.588\ \text{tons/hr}^2$ This means that 5 hours into the workday, the rate at which the gravel is arriving is decreasing.
- This is just the **net change of gravel**. Because $G(t)$ is the rate function, net change is $\int_0^8 G(t)dt = 825.551$ tons
- This is the first part to use our accumulation function. To know if the gravel is increasing or decreasing, I need to know if the rate is positive or negative. $A(t) = G(t) - R(t)$ is already the rate. $A(5) = G(5) - R(5) = 98.141 - 100 = -1.859$ tons/hour. The amount of gravel is **decreasing**.
- To find the maximum amount
  - we need to find the turning points compare with the end points. The turning points occur when $A(t) = 0$ or rather when $G(t) = R(t)$
    - $G(t) = R(t)$ is the same as $G(t) = 100$
    - Using a calculatron, this is true only at $t = 4.923$
  - To find the amount at $t = 0, 4.923$ and 8, we need to integrate. While integrating, we note that at $t = 0$ there are 500 tons.
    - At $t = 8$, $500 + \int_0^8 A(t)dt = 525.551$ tons
    - At $t = 4.923$, $500 + \int_0^{4.923} A(t)dt = 635.376$ tons
  - From here, we can conclude the maximum occurs at $t = 4.923$

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Example

The rate at which rainwater flows through a drainpipe is modeled by the function R, where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \le t \le 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \le t \le 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

- How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \le t \le 8$?
- Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.
- At what time t, $0 \le t \le 8$, is the amount of water in the pipe at a maximum? Justify your answer.
- The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

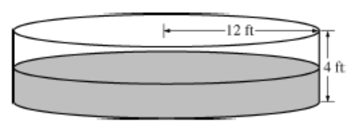
Answer

Our rate of accumulation will be $A(t) = R(t) - D(t)$. Since both of these are equations, we can even write more simply $R(t) = 20\sin\left(\frac{t^2}{35}\right) + 0.04t^3 - 0.4t^2 + 0.96t$ cubic feet per hour.

- $\int_0^8 R(t)dt = 76.570$ cubic feet
- $A(3) = R(3) - D(3) = -0.313$ cubic feet per hour. Because the rate of accumulation is negative, the amount of water is decreasing.
- $A(t) = 0$ at $t = 3.232$. At $t = 0$, the pipe has 30 cubic feet. At $t = 8$, the pipe has $30 + \int_0^8 A(t)dt = 48.544$ cubic feet. At $t = 3.232$, the pipe has $30 + \int_0^{3.232} A(t)dt = 27.966$ cubic feet. The maximum is at $t = 8$
- $50 = 30 + \int_0^w \big(R(t) - D(t)\big)dt$ Note that I didn't use $A(t)$ here because ultimately, I want my answers to include only the symbols used by the problem.

Practice

t	0	2	4	6	8	10	12
$P(t)$	0	46	53	57	60	62	63



1.) The figure above shows an aboveground swimming pool in the shape of a cylinder with a radius of 12 feet and a height of 4 feet. The pool contains 1000 cubic feet of water at time $t = 0$. During the time interval $0 \le t \le 12$ hours, water is pumpted into the pool at the rate $P(t)$ cubic feet per hour. The table above gives values of $P(t)$ for selected values of t. During the same time interval, water is leaking from the pool at the rate $R(t)$ cubic feet per hour, where $R(t) = 25e^{-0.05t}$ (Note: the volume V of a cylinder with radius r and height h is given by $V = \pi r^2 h$.)

a. Use a midpoint Riemann sum with three subintervals of equal length to approximate the total amount of water that was pumped into the pool during the time interval $0 \le t \le 12$ hours. Show the computations that lead to your answer.

b. Calculate the total amount of water that leaked out of the pool during the time interval $0 \le t \le 12$ hours.

c. Use the results from part a) and b) to approximate the value of water in the pool at time $t = 12$ hours.

d. Find the rate at which the volume of water in the pool is increase at time $t = 8$ hours.

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2.) There is no snow on Mitya's driveway when snow begins to fall at midnight. From midnight to 9am, snow accumulates on the driveway at a rate modeled by $f(t) = 7te^{\cos t}$ cubic feet per hour, where t is measured in hours since midnight. Mitya starts removing snow at 6am ($t = 6$). The rate $g(t)$, in cubic feet per hour, at which Mitya removes snow from the driveway at time t hours after midnight is modeled by

$$g(t) = \begin{cases} 0 & \text{for} \quad 0 \le t < 6 \\ 125 & \text{for} \quad 6 \le t < 7 \\ 108 & \text{for} \quad 7 \le t \le 9 \end{cases}$$

a. How many cubic feet of snow have accumulated on the driveway by 6am?
b. Find the rate of change of the volume of snow on the driveway at 8am?
c. Let $h(t)$ represent the total amount of snow, in cubic feet, that Mitya has removed from the driveway at time t hours after midnight. Express h as a piecewise defined function with domain $0 \le t \le 9$
d. How many cubic feet of snow are on the driveway at 9am?

t (hours)	0	2	5	7	8
$E(t)$ (hundreds of entries)	0	4	13	21	23

3.) A zoo sponsored a one-day contest to name a new baby elephant. Zoo visitors deposited entries in a special box between noon ($t = 0$) and 8pm ($t = 8$). The number of entries in the box t hours after noon is modeled by a differentiable function E for $0 \le t \le 8$. Values of $E(t)$ in hundreds of entries, at various times t are shown in the table above.

a. Use the data in the table to approximate the rate, in hundreds of entries per hour, at which entries were being deposited at time $t = 6$. Show the computations that lead to your answer.
b. Use a trapezoidal sum with the four subintervals given by the table to approximate the value of $\frac{1}{8}\int_0^8 E(t)dt$. Using correct units, explain the meaning of $\frac{1}{8}\int_0^8 E(t)dt$ in terms of the number of entries.
c. At 8pm, the volunteers began to process the entries. They processed the entries at a rate modeled by the function P, where $P(t) = t^3 - 30t^2 + 298t - 976$ hundreds of entries per hour for $8 \le t \le 12$. According to the model, how many entries had not yet been processed by midnight ($t = 12$)?
d. According to the model from part c), at what time were the entries being processed most quickly? Justify your answer.

4.) The penguin population on an island is modeled by a differentiable function P of time t, where $P(t)$ is the number of penguins and t is measured in years, for $0 \le t \le 40$. There are 100,000 penguins on the island at time $t = 0$. The birth rate for the penguins on the island is modeled by $B(t) = 1000e^{0.06t}$ penguins per year and the death rate for the penguins on the island is modeled by $D(t) = 250e^{0.1t}$ penguins per year.

a. What is the rate of change of the penguin population on the island at time $t = 0$?
b. What is the penguin population on the island at time $t = 40$?
c. What is the average rate of change of the penguin population on the island for $0 \le t \le 40$?
d. To the nearest whole number, find the absolute minimum penguin population and the absolute maximum penguin population on the island for $0 \le t \le 40$. Show the analysis that leads to your answers.

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Last Page! Perfect For Cat Pictures!